Math 249 Lecture 32 Notes

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1 Chambers and chamber walks

If G is a Coxeter group, we have associated to it a hyperplane arrangement $\{H : s_H \in G\}$. We can look at $\mathbb{R}^n \setminus \bigcup H$, which is a disjoint union of chambers. Their closures are called *Weyl chambers*.

In this lecture, we will be using some arguments that appeal to intuition in 3-dimensions but in fact generalize to n dimensions. For a more formal treatment, check the book *Reflection Groups and Coxeter Groups* by James Humphreys.

1.1 Normal vectors to chamber walls

Pick a chamber C, where the walls are hyperplanes H_i. Each chamber has a normal vector α_i, where we can choose α_i to point to the inside the chamber. This is the condition that $\langle \alpha_i, v\rangle \geq 0$ for every vector $v \in C$. If we let H_v to be the hyperplane normal to v, we get that all the α_i are on one side of H_v.

There exists some cross-section of the space such that any two walls of C look like two intersecting lines, intersecting at the origin. In fact, this is true for any number of walls of a chamber. We then get a 2-dimensional picture of how the chamber looks on this slice. The angle between any two hyperplanes must be 90 degrees or less, so we get that $\langle \alpha_i, \alpha_j\rangle \leq 0$.

Definition 1.1. A set of vectors v_i is $\mathbb{R}^+$ *independent* if any positive linear combination of them is nonzero.

The α_i are then $\mathbb{R}^+$ independent. We can now make use of the following fact.

Lemma 1.1. *If $\alpha_1, \dots, \alpha_k$ are $\mathbb{R}^+$ independent and $\langle \alpha_i, \alpha_j\rangle \leq 0$, then the α_i are $\mathbb{R}$-linearly independent.*

Proof. Suppose that $\sum c_i\alpha_i = 0$ with not all $c_i = 0$. We must have some $c_i > 0$ and some $c_i < 0$. Moving the negative terms to the other side, we get

$$\sum c_i\alpha_i = \sum d_j\alpha_j$$

with $c_i, d_j > 0$ and $i \neq j$. Call this quantity w. Then

$$\langle w, w \rangle = \sum_{i,j} \underbrace{c_i d_j}_{>0} \underbrace{\langle \alpha_i, \alpha_j \rangle}_{\leq 0} \leq 0.$$

Since $\langle w, w \rangle \geq 0$ for any inner product and vector w, we get $\langle w, w \rangle\, 0$, which means $w = 0$. This is a contradiction. $\square$

Corollary 1.1. *The normal vectors α_i are linearly independent; i.e. C is simplicial (or degenerate simplicial).*

1.2 Coxeter group action on chambers

The group G acts on the hyperplane arrangement, so we want to show that it acts on the chambers.

Definition 1.2. Let G be a Coxeter group and C a chamber. The *simple reflections* s_i are the reflections s_{H_i}, where H_i are the hyperplanes that are the walls of C.

For a choice of C, let $\tilde{G} \subseteq G$ be the subgroup generated by the simple reflections.

Proposition 1.1. $\bigcup \tilde{G} \cdot C = \mathbb{R}^n$.

Proof. This is a finite union of closed sets, so it is closed. It is also nonempty. We want to show that this is open. It suffices to show that it contains a neighborhood of every $v \in C$. If v is in the interior of C, we are done. If it is on a wall, take a chamber with dimension $n-1$ on this hyperplane; by induction, we can find a neighborhood on the plane, and we can use $\tilde{G}$ to get a neighborhood of v. $\square$

Corollary 1.2. $\tilde{G} = G$.

Proof. For each $s_H \in G$, there is some $g \in \tilde{G}$ such that $s_H = g_{s_i} := g s_i g^{-1} \in \tilde{G}$. So the simple reflections s_i generate G. $\square$

Example 1.1. Consider $S_n \circlearrowright \mathbb{R}^n$, where $(i\ j) = s_H$, where $H = \{x \in \mathbb{R}^n : x_i = x_j\}$. Consider $C = \{x \in \mathbb{R}^n : x_1 > x_2 > \cdots > x_n\}$; this is a chamber. The walls of C are $\{x \in \mathbb{R}^n : x_i = x_{i+1}\}$ for each i. The simple reflections are $(i\ (i+1))$. The chambers are wC for $w \in S_n$.

Example 1.2. Consider $B_n \circlearrowright \mathbb{R}^n$. The reflecting hyperplanes are $\{x \in \mathbb{R}^n : x_i = x_j\}$ and $\{x \in \mathbb{R}^n : x_i = 0\}$. Let $C = \{x \in \mathbb{R}^n : x_1 > \cdots > x_n > 0\}$. The walls are of the form $\{x \in \mathbb{R}^n : x_i = x_{i+1}\}$ and $\{x \in \mathbb{R}^n : x_n = 0\}$. The simple reflections are $(i\ (i+1))$, like with S_n, but we also have the reflection $(\bar{i})$.

1.3 Chamber walks

Pick a chamber C, and look at one of its walls, corresponding to the reflection s_{i_1}. Look at $s_{i_1}C$, an adjacent chamber where we have reflected across the wall. We can keep reflecting, by looking at $s_{i_1}(s_{i_2}C)$. Make sure you check this; when you reflect over a wall, the element gets multiplied on the right, not the left. So a chamber walk from C to wC will correspond to a factorization $w = s_{i_1} \cdots s_{i_\ell}$ into simple reflections.

Define $\ell(w)$ to be the minimum ℓ such that $w = s_{i_1} \cdots s_{i_\ell}$; this is the minimum length of a chamber walk from C to wC. Every simple reflection will make the walk cross over a hyperplane. So $\ell(w) \geq$ the number of hyperplanes H separating C from wC.

Example 1.3. Let $w \in S_n$ and $C = \{x \in \mathbb{R}^n : x_1 > \cdots > x_n\}$. Then

$$w \cdot (x_1, \ldots, x_n) = (x_{w^{-1}(1)}, \ldots, x_{w^{-1}(n)}).$$

So in the case of S_n, the number of hyperplanes H separating C from wC is the number of inversions of w^{-1} (or of w).